

Propositional Logic

Horn Formulas

Efficient satisfiability checks

In the following:

- ▶ A very efficient satisfiability check for the special class of **Horn formulas**.
- ▶ Efficient satisfiability checks for arbitrary formulas in CNF: **resolution** (later).

Horn formulas

Definition

A formula F in CNF is a **Horn formula** if every disjunction in F contains at most one positive literal.

A disjunction in a Horn formula can equivalently be viewed as an implication $K \rightarrow B$ where K is a conjunction of atoms or $K = \top$ and B is an atom or $B = \perp$:

$$\begin{aligned}(\neg A \vee \neg B \vee C) &\equiv (A \wedge B \rightarrow C) \\(\neg A \vee \neg B) &\equiv (A \wedge B \rightarrow \perp) \\A &\equiv (\top \rightarrow A)\end{aligned}$$

Satisfiability check for Horn formulas

Input: a Horn formula F .

Algorithm building a model (assignment) \mathcal{M} :

```
for all atoms  $A_i$  in  $F$  do  $\mathcal{M}(A_i) := 0$ ;  
while  $F$  has a subformula  $K \rightarrow B$   
    such that  $\mathcal{M}(K) = 1$  and  $\mathcal{M}(B) = 0$   
do  
    if  $B = \perp$  then return “unsatisfiable”  
    else  $\mathcal{M}(B) := 1$   
return “satisfiable”
```

Maximal number of iterations of the while loop:
number of implications in F

Each iteration requires at most $O(|F|)$ steps.

Overall complexity: $O(|F|^2)$

[Algorithm can be improved to $O(|F|)$. See Schönig.]

Correctness of the model building algorithm

Theorem

The algorithm returns “satisfiable” iff F is satisfiable.

Proof Observe: if the algorithm sets $\mathcal{M}(B) = 1$, then $\mathcal{A}(B) = 1$ for every assignment \mathcal{A} such that $\mathcal{A}(F) = 1$. This is an invariant.

(a) If “unsatisfiable” then unsatisfiable.

We prove unsatisfiability by contradiction. Assume $\mathcal{A}(F) = 1$ for some \mathcal{A} . Let $(A_{i_1} \wedge \dots \wedge A_{i_k} \rightarrow \perp)$ be the subformula causing “unsatisfiable”. Since $\mathcal{M}(A_{i_1}) = \dots = \mathcal{M}(A_{i_k}) = 1$, $\mathcal{A}(A_{i_1}) = \dots = \mathcal{A}(A_{i_k}) = 1$. Then $\mathcal{A}(A_{i_1} \wedge \dots \wedge A_{i_k} \rightarrow \perp) = 0$ and so $\mathcal{A}(F) = 0$, contradiction. So F has no satisfying assignments.

(b) If “satisfiable” then satisfiable.

After termination with “satisfiable”, for every subformula $K \rightarrow B$ of F , $\mathcal{M}(K) = 0$ or $\mathcal{M}(B) = 1$. Therefore $\mathcal{M}(K \rightarrow B) = 1$ and thus $\mathcal{M} \models F$. In fact, the invariant shows that \mathcal{M} is the **minimal** model of F .