

# First-order Predicate Logic Theories

# Definitions

## Definition

A **signature**  $\Sigma$  is a set of predicate and function symbols.

A  **$\Sigma$ -formula** is a formula that contains only predicate and function symbols from  $\Sigma$ .

A  **$\Sigma$ -structure** is a structure that interprets all predicate and function symbols from  $\Sigma$ .

## Definition

A **sentence** is a closed formula.

In the sequel,  $S$  is a set of sentences.

# Theories

## Definition

A *theory* is a set of sentences  $S$  such that  $S$  is closed under consequence: If  $S \models F$  and  $F$  is closed, then  $F \in S$ .

Let  $\mathcal{A}$  be a  $\Sigma$ -structure:

$Th(\mathcal{A})$  is the set of all sentences true in  $\mathcal{A}$ :

$$Th(\mathcal{A}) = \{F \mid F \text{ } \Sigma\text{-sentence and } \mathcal{A} \models F\}$$

## Lemma

Let  $\mathcal{A}$  be a  $\Sigma$ -structure and  $F$  a  $\Sigma$ -sentence.

Then  $\mathcal{A} \models F$  iff  $Th(\mathcal{A}) \models F$ .

## Corollary

$Th(\mathcal{A})$  is a theory.

## Example

**Notation:**  $(\mathbb{Z}, +, \leq)$  denotes the structure with universe  $\mathbb{Z}$  and the standard interpretations for the symbols  $+$  and  $\leq$ .

The same notation is used for other standard structures where the interpretation of a symbol is clear from the symbol.

### Example (Linear integer arithmetic)

$Th(\mathbb{Z}, +, \leq)$  is the set of all sentences over the signature  $\{+, \leq\}$  that are true in the structure  $(\mathbb{Z}, +, \leq)$ .

# Axioms and consequences

## Definition

Let  $S$  be a set of  $\Sigma$ -sentences.

$Cn(S)$  is the set of **consequences** of  $S$ :

$$Cn(S) = \{F \mid F \text{ } \Sigma\text{-sentence and } S \models F\}$$

A theory  $T$  is **axiomatized** by  $S$  if  $T = Cn(S)$

A theory  $T$  is **axiomatizable** if there is some decidable or recursively enumerable  $S$  that axiomatizes  $T$ .

A theory  $T$  is **finitely axiomatizable**

if there is some finite  $S$  that axiomatizes  $T$ .

## Examples

$Cn(\emptyset)$  is the set of valid sentences.

$Cn(\{\forall x \forall y \forall z (x * y) * z = x * (y * z)\})$  is the set of sentences that are true in all semigroups.

## Famous numerical theories

$Th(\mathbb{R}, +, \leq)$  is called **linear real arithmetic**.

It is decidable.

$Th(\mathbb{R}, +, *, \leq)$  is called **real arithmetic**.

It is decidable.

$Th(\mathbb{Z}, +, \leq)$  is called **linear integer arithmetic** or **Presburger arithmetic**.

It is decidable.

$Th(\mathbb{Z}, +, *, \leq)$  is called **integer arithmetic**.

It is not even **semidecidable** (= r.e.).

Decidability via special algorithms.

# Completeness and elementary equivalence

## Definition

A theory  $T$  is **complete** if for every sentence  $F$ ,  $T \models F$  or  $T \models \neg F$ .

## Definition

Two structures  $\mathcal{A}$  and  $\mathcal{B}$  are **elementarily equivalent** if  $Th(\mathcal{A}) = Th(\mathcal{B})$ .

## Theorem

*A theory  $T$  is complete iff all its models are elementarily equivalent.*