Logics Exercise

TU München
Institut für Informatik

Prof. Tobias Nipkow
Lars Hupel

SS 2017 — Exercise Sheet 7 — 13.06.2017

Submission of homework: Before tutorial on 20.06.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 7.1. [Natural Numbers and FOL]
We consider the following axioms in an attempt to model the natural numbers in predicate logic:

1. $F_1 = \forall x \forall y (f(x) = f(y) \rightarrow x = y)$
2. $F_2 = \forall x (f(x) \neq 0)$
3. $F_3 = \forall x (x = 0 \vee \exists y (x = f(y)))$

Give a model with an *uncountable* universe for:

1. $\{F_1, F_2\}$
2. $\{F_1, F_2, F_3\}$

Hint: A set S is uncountable if there is no bijection between S and $\mathbb{N}$.

Exercise 7.2. [Occurs Check]
What happens if one omits the occurs check in the unification algorithm? Find an example where a unification algorithm without occurs check diverges or returns the wrong result.

Exercise 7.3. [Unifiable Terms]
Specify the most general unifiers for the following sets of terms, if one exists:

$$L_1 = \{f(x, y), f(h(a), x)\}$$
$$L_2 = \{f(x, y), f(h(x), x)\}$$
$$L_3 = \{f(x, b), f(h(y), z)\}$$
$$L_4 = \{f(x, x), f(h(y), y)\}$$

Exercise 7.4. [Formulas without Negation]
Prove that every predicate logic formula that only contains $\wedge, \vee, \forall, \exists, \longrightarrow$ and atomic formulas is satisfiable. Is such a formula also valid?

Homework 7.1. **[(In)finite Models]** (6 points)

1. Show that any model (for a formula of predicate logic) with an universe of size n can be extended to a model of size m for any $m \geq n$. Can it also be extended to an *infinite* model?
2. Now consider the extension of predicate logic with equality. Does above property still hold?

Homework 7.2. **[Simultaneous Substitution]** (6 points)
Recall that $F[t_1/x_1, \ldots, t_n/x_n]$ is the *simultaneous* substitution of $x_1, \ldots, x_n$ by $t_1, \ldots, t_n$.

1. Can we always express $F[t_1/x_1, \ldots, t_n/x_n]$ as a series of one-variable substitutions?
2. Can we always summarize a series of one-variable substitutions to a single simultaneous substitution?

Homework 7.3. **[Most general unifier]** (6 points)
Consider the unification problem $x \stackrel{?}{=} y$. Without running the unification algorithm, prove that

1. $\sigma_1 = \{x \mapsto y\}$ is a most general unifier.
2. $\sigma_2 = \{x \mapsto z, y \mapsto z\}$ is unifier, but not a most general unifier.

Hint: Argue using the definition of "most general unifier". Two substitutions σ and σ' can be proven equal by showing that they are equal on all variables, i.e., for all x, $x\sigma = x\sigma'$. Similarly, they can be proven unequal by demonstrating for a particular x that $x\sigma \neq x\sigma'$.

Homework 7.4. **[Unification]** (2 points)
Use the algorithm presented in the lecture to compute a most general unifier for the following set of formulas: $\{P(x, y), P(f(a), g(x)), P(f(z), g(f(z)))\}$