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Submission of homework: Before tutorial on 18.07.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 11.1. [Decidable Theories]
Let S be a set of sentences (i.e. closed formulas) such that S is closed under consequence: if $S \models F$ and F is closed, then $F \in S$. Additionally, assume that S is finitely axiomatizable and complete, i.e. $F \in S$ or $\neg F \in S$ for any sentence F.

1. Give a procedure for deciding whether $S \models F$ for a sentence F.
2. Can you obtain a similar result when the assumption is that the axiom system is only *recursively enumerable*?

Exercise 11.2. [QE for DLO]
Use the quantifier-elimination procedure for DLOs to check whether the following formula is a member of $Th(DLO)$:

$$\exists x \forall y \exists z ((x < y \vee z < x) \wedge y < z)$$

Exercise 11.3. [Consequence]
Show that Cn is a closure operator, i.e. Cn fulfills the following properties:

- $S \subseteq Cn(S)$
- if $S \subseteq S'$ then $Cn(S) \subseteq Cn(S')$
- $Cn(Cn(S)) = Cn(S)$

Exercise 11.4. [Axiomatizations and Compactness]
The compactness theorem for first-order logic: *If every finite subset of a (countable) set M of formulas has a model, then M as a whole has a model.*

Using compactness, show that if a theory is finitely axiomatizable, any countable axiomatization of it has a finite subset that axiomatizes the same theory. In other words, if $Cn(\Gamma) = Cn(\Delta)$ with Γ countable and Δ finite, then there is a finite $\Gamma' \subseteq \Gamma$ with $Cn(\Gamma') = Cn(\Gamma)$.

Homework 11.1. [QE for DLO] (8 points)
Use the quantifier-elimination procedure for DLOs to check whether the following formulas are member of $Th(DLO)$:

1. $\forall x \forall y \forall z (x < y \rightarrow (y < z \rightarrow x < z))$
2. $\exists x \exists y \forall z ((z < x \rightarrow z \leq y) \wedge (z > y \rightarrow z \geq x))$

Hint: Assume that $\rightarrow$ is defined in terms of $\vee$ and $\neg$; $\leq$ in terms of $=$, $<$ and $\vee$ (and similar for $>$ and $\geq$).

Homework 11.2. [Refining Fourier-Motzkin] (6 points)
Show how Fourier-Motzkin elimination can be extended to directly handle constraints of the form $x \leq y$ instead of rewriting them to $x < y \vee x = y$ first.

Homework 11.3. [Difference Logic] (6 points)
We consider a fragment of linear arithmetic, in which atomic formulas only take the form $x - y \leq c$ for variables x and y, and $c \in \mathbb{Q}$.

For a finite set S of such difference constraints, we can define a corresponding inequality graph $G(V, E)$, where V is the set of variables of S, and E consists of all the edges (x, y) with weight c for all constraints $x - y \leq c$ of S. Show that the conjuction of all constraints from S is satisfiable iff G does not contain a negative cycle.

How can you use this theorem to obtain a procedure for deciding wether a formula is a member of this fragment?