Logics Exercise

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Submission of homework: Before tutorial on 10.05.2017. Until further notice, homework has to be submitted in groups of two students.
You have to achieve a minimum combined score of 50% on your homeworks to be granted a 0.3/0.4 grade bonus on the final exam, provided that the exam is passed.

Exercise 1.1. [Short Questions]
Let M be a set of formulas, and let F and G be formulas. Which of the following assertions hold?

1. If F satisfiable then $M \models F$
2. F is valid iff $\top \models F$
3. If $\models F$ then $M \models F$
4. If $M \models F$ then $M \cup \{G\} \models F$
5. $M \models F$ and $M \models \neg F$ cannot hold simultaneously
6. If $M \models G \to F$ and $M \models G$ then $M \models F$

Solution:
The assertions 2, 3, 4, and 6 hold.

Counterexample for 1: $F = A_1, M = \{A_2\}$

Counterexample for 5: $M = \{\perp\}$ *(ex falso quodlibet)*

Exercise 1.2. [Coincidence Lemma]
Assume that for all atomic formulas A_i in F, $\mathcal{A}(A_i) = \mathcal{A}'(A_i)$. Show that

$$\mathcal{A} \models F \text{ iff } \mathcal{A}' \models F$$

Solution:
Proof by induction over the structure of F. Let $atoms(F)$ denote the set of all atomic formulas A_i in a formula F.

- Base case $F = A_i$ for some i:
  Observation: $atoms(A_i) = \{A_i\}$, hence $\mathcal{A}(A_i) = \mathcal{A}'(A_i)$
  $\mathcal{A} \models A_i$ iff $\mathcal{A}(A_i) = 1$ iff $\mathcal{A}'(A_i) = 1$ iff $\mathcal{A}' \models A_i$
- Base case $F = \top$: trivial
- Base case $F = \bot$: trivial
- Case $F = \neg G$ for some G:
  Observation: $atoms(\neg G) = atoms(G)$
  IH: $\mathcal{A} \models G$ iff $\mathcal{A}' \models G$
  $\mathcal{A} \models \neg G$ iff $\mathcal{A} \not\models G$ iff $\mathcal{A}' \not\models G$ iff $\mathcal{A}' \models \neg G$
- Case $F = G \vee H$ for some G, H:
  Observation: $atoms(F) = atoms(G) \cup atoms(H)$
  Hence, $\mathcal{A}$ and $\mathcal{A}'$ coincide on G and H too.
  IH 1: $\mathcal{A} \models G$ iff $\mathcal{A}' \models G$
  IH 2: $\mathcal{A} \models H$ iff $\mathcal{A}' \models H$
  Remaining proof trivial.
- Case $F = G \wedge H$ and remaining cases: similar

Exercise 1.3. [Semantic Proof]
Let $\models F \rightarrow G$ where F and G do not share any atoms. Show that then F is unsatisfiable or G is a tautology (or both). *Hint:* you may want to use the previous result.

Solution:
Proof by contradiction. Assume that F is satisifiable *and* G is not a tautology. Obtain assignments $\mathcal{A}_F$ and $\mathcal{A}_G$ such that $\mathcal{A}_F \models F$ and $\mathcal{A}_G \not\models G$. Construct a new assignment $\mathcal{A}$ as follows:

$$\mathcal{A}(A_i) = \begin{cases} \mathcal{A}_F(A_i) & \text{if } A_i \in atoms(F) \\ \mathcal{A}_G(A_i) & \text{if } A_i \in atoms(G) \\ 0 & \text{otherwise} \end{cases}$$

This is well-defined, because $atoms(F) \cap atoms(G) = \emptyset$. $\mathcal{A}$ coincides with $\mathcal{A}_F$ on F and with $\mathcal{A}_G$ on G. By coincidence lemma, $\mathcal{A} \models F$ and $\mathcal{A} \not\models G$. But $\mathcal{A} \not\models (F \rightarrow G)$, which is a contradiction to $\models F \rightarrow G$.

Exercise 1.4. **[Satisfiability Algorithms]**
Check the following formulas for satisfiability using one of the algorithms seen in the lecture:

- $F_1 = (\neg A \vee \neg D \vee B) \wedge D \wedge \neg B \wedge E \wedge (\neg D \vee \neg E \vee C)$
- $F_2 = (A \to C) \wedge (C \to E) \wedge (\top \to B) \wedge (C \wedge A \to D) \wedge (B \to A) \wedge$
  $(B \to E) \wedge (D \wedge E \to \bot)$
- $F_3 = (A \to E) \wedge (B \to \bot) \wedge (C \to B) \wedge (\top \to A) \wedge (E \wedge B \to C) \wedge (C \to D)$

Solution:

- Rewrite to $(A \wedge D \to B) \wedge (\top \to D) \wedge (B \to \bot) \wedge (\top \to E) \wedge (D \wedge E \to C)$. Mark D, E, C. Satisfiable.
- Mark B, A, E, C, D. Unsatisfiable.
- Mark A, E. Satisfiable.

Homework 1.1. **[CNF and DNF]** (6 points)
Use the rewriting-based procedure from the lecture to convert the following formulas F and G first to NNF, and then to CNF and DNF. Document each rewriting step.

$$F = \neg\neg(\neg A_1 \wedge \neg\neg(A_2 \vee A_3)) \qquad G = (A_1 \vee A_2 \vee A_3) \wedge (\neg A_1 \vee \neg A_2)$$

Homework 1.2. **[Basic equivalences]** (8 points)
Let F and G be formulas. Are the following statements equivalent? Proof or counterexample!

1. $\models F \leftrightarrow G$
2. $F \equiv G$

How about these two statements?

1. F is valid
2. $F \equiv \top$

Homework 1.3. **[Efficient CNF satisfiability check]** (6 points)
In general, solving satisfiability for CNF formula is a hard problem. Consider the special case where clauses may only contain up to two literals. Give an efficient algorithm to check satisfiability.