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SS 2017 | Exercise Sheet 3 | 16.05.2017

Submission of homework: Before tutorial on 23.05.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 3.1. [**System G1c**]
An alternative definition of the sequent calculus ("G1c") is defined as follows:

Axioms

$$\text{Ax}\quad A \Rightarrow A \qquad\qquad \text{L}\bot\quad \bot \Rightarrow$$

Rules for weakening (W) and contraction (C)

$$\text{LW}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad\qquad \text{RW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$\text{LC}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad\qquad \text{RC}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

Rules for the logical operators

$$\text{L}\wedge\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta}\ (i = 0, 1) \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1}\ (i = 0, 1)$$

$$\text{L}\to\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \text{R}\to\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

Notably, weaking and contraction are built-in rules. Show that sequent calculus can be simulated by G1c, i.e., $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_{G1c} \Gamma \Rightarrow \Delta$.

Solution:

Exercise 3.1

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta}\ \wedge L$$

simulated by:

$$\dfrac{\dfrac{\dfrac{F, G, \Gamma \Rightarrow \Delta}{F, F \wedge G, \Gamma \Rightarrow \Delta}\ L\wedge 1}{F \wedge G, F \wedge G, \Gamma \Rightarrow \Delta}\ L\wedge 0}{F \wedge G, \Gamma \Rightarrow \Delta}\ LC$$

$$\frac{F, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg F, \Delta}\ \neg R$$

Simulated by:

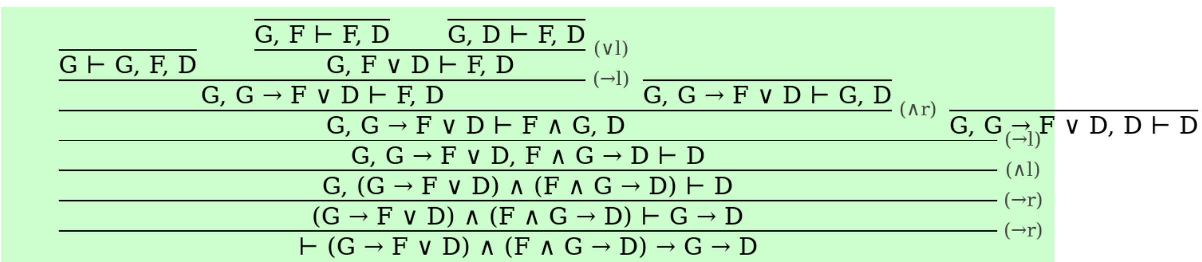
$$\dfrac{\dfrac{\dfrac{F, \Gamma \Rightarrow \Delta}{F, \Gamma \Rightarrow \bot, \Delta}\ RW}{\Gamma \Rightarrow (F \to \bot), \Delta}\ R\to}{\Gamma \Rightarrow \neg F, \Delta}\ \text{unfold } \neg$$

Exercise 3.2. **[Cut Elimination, Semantically]**
Semantically prove the admissibility of cut elimination.

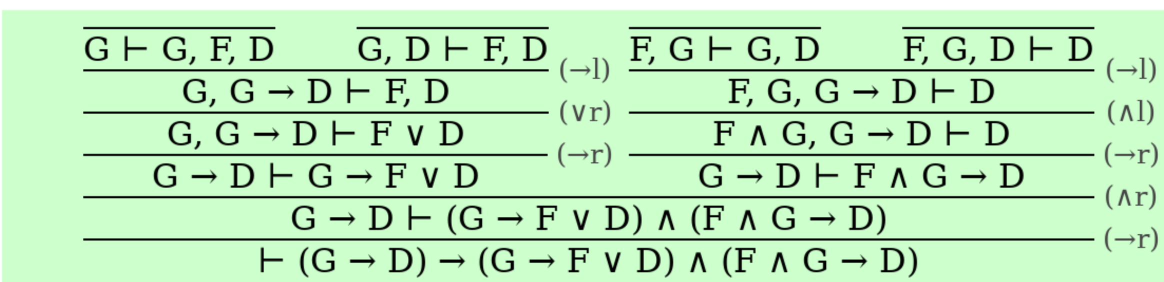
Solution:
Recall cut elimination from the lecture:

$$\text{If } \vdash_G \Gamma \Rightarrow F, \Delta \text{ and } \vdash_G F, \Gamma \Rightarrow \Delta \text{ then } \vdash_G \Gamma \Rightarrow \Delta$$

To prove this semantically, we have to show that given $|\Gamma \Rightarrow F, \Delta|$ and $|F, \Gamma \Rightarrow \Delta|$, $|\Gamma \Rightarrow \Delta|$ holds. In this case, an even stronger property holds: precedent and antecedent are equivalent. That is, $(G \to F \vee D) \wedge (F \wedge G \to D) \equiv G \to D$. We can prove this with sequent calculus:[1]



The other direction is similar:[2]



[1] http://logitext.mit.edu/proving/+.28G+.2D.3E+F+.5C.2F+D.29+.2F.5C+.28F+.2F.5C+G+.2D.3E+D.29+.2D.3E+.28G+.E2.86.92+D.29

[2] http://logitext.mit.edu/proving/.28G+.2D.3E+D.29+.2D.3E+.28.28G+.2D.3E+F+.5C.2F+D.29+.2F.5C+.28F+.2F.5C+G+.2D.3E+D.29.29

Exercise 3.3. [**Atomic Cut**]
Let A be an atomic formula. Prove that if $\vdash_G \Gamma \Rightarrow A, \Delta$ and $\vdash_G A, \Gamma \Rightarrow \Delta$, then $\vdash_G \Gamma \Rightarrow \Delta$.

Solution:
Let D_1 and D_2 be the derivations of the assumptions. Proof by induction on the depth of D_1. Case analysis: What is the last proof rule in D_1?

Ax There are two subcases:

1. A is the principal formula: $D_1 = \dfrac{}{\underbrace{A, \Gamma'}_{\Gamma} \Rightarrow A, \Delta}$

   Contract D_2: $A, \Gamma' \Rightarrow \Delta$
2. A is not principal: $D_1 = \dfrac{}{\underbrace{B, \Gamma'}_{\Gamma} \Rightarrow A, \underbrace{B, \Delta'}_{\Delta}}$

   $\vdash_G \Gamma \Rightarrow \Delta$ by Ax

$\wedge$**L** $D_1 = \dfrac{F, G, \Gamma' \overset{D_1'}{\Rightarrow} A, \Delta}{\underbrace{F \wedge G, \Gamma'}_{\Gamma} \Rightarrow A, \Delta}$

$$\dfrac{\dfrac{D_1' \qquad \dfrac{D_2}{A, F, G, \Gamma' \Rightarrow \Delta}\ \wedge\mathrm{L}^{-1}}{F, G, \Gamma' \Rightarrow \Delta}\ \mathrm{IH}}{\Gamma \Rightarrow \Delta}\ \wedge\mathrm{L}$$

Other cases are similar.

Exercise 3.4. [**More Connectives**]
Define sequent rules for the logical connectives "nand" ($\overline{\wedge}$) and "xor" ($\otimes$). **Solution:**

Exercise 3.4

$$\frac{\Gamma \Rightarrow \Delta, p \qquad \Gamma \Rightarrow \Delta, q}{\Gamma, p \barwedge q \Rightarrow \Delta} \; \barwedge L$$

$$(\Gamma \rightarrow \Delta \vee p) \wedge (\Gamma \rightarrow \Delta \vee q)$$

$$\equiv ?$$

$$(\Gamma \wedge \underbrace{(p \barwedge q)}_{\neg(p \wedge q)}) \rightarrow \Delta$$

$$\frac{\Gamma, p, q \not\Rightarrow \Delta}{\Gamma \Rightarrow \Delta, p \barwedge q} \; \barwedge R$$

$$\frac{\Gamma, p \Rightarrow \Delta, q \qquad \Gamma, q \Rightarrow \Delta, p}{\Gamma, \underbrace{p \otimes q}_{(p \wedge \neg q) \vee (\neg p \wedge q)} \Rightarrow \Delta} \; \otimes L$$

$$\frac{\Gamma \Rightarrow \Delta, p, q \qquad \Gamma, p, q \Rightarrow \Delta}{\Gamma \not\Rightarrow \Delta, p \otimes q} \; \otimes R$$

Homework 3.1. [Intermediate Formulas] (6 points)
Let F, G be formulas such that $F \models G$. Prove that there is an *intermediate formula* H such that the following three conditions hold:

1. H contains only atomic formulas that occur in both F and G
2. $F \models H$
3. $H \models G$

How can H be constructed?

Homework 3.2. [Sequent Calculus] (2 points)
Prove the formula $((A_1 \to A_2) \to A_1) \to A_1$ in System G1c.

Homework 3.3. [Inversion Rules] (6 points)
Show that the following inversion rules are admissible:

$$\frac{F_1 \vee F_2, \Gamma \Rightarrow \Delta}{F_1, \Gamma \Rightarrow \Delta} \qquad \frac{F_1 \vee F_2, \Gamma \Rightarrow \Delta}{F_2, \Gamma \Rightarrow \Delta}$$

Homework 3.4. [Lop-Sided Sequent Calculus] (6 points)
In sequent calculus, each sequent consists of an antecedent and a consequent: $\vdash_G \Gamma \Rightarrow \Delta$. But it turns out that either side is unneeded. Define an inference system in which the antecedent (left-hand side) is always empty. Consider the following points in your design:

- An "old" sequent $\vdash_G \Gamma \Rightarrow \Delta$ should correspond to a "new" sequent $\vdash_{G'} \{\} \Rightarrow \neg\Gamma, \Delta$. Note that Γ is a conjunction which turns into a disjunction in the consequent (right-hand side).
- For notational convenience, you may just use Δ instead of $\{\} \Rightarrow \Delta$.
- Sketch your idea and why your design is correct. The new system should be able to simulate the old one and vice versa. Pick one rule in each system and explain how they can be simulated in the other one.