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SS 2017 | Exercise Sheet 7 | 13.06.2017

Submission of homework: Before tutorial on 20.06.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 7.1. [Natural Numbers and FOL]
We consider the following axioms in an attempt to model the natural numbers in predicate logic:

1. $F_1 = \forall x \forall y (f(x) = f(y) \rightarrow x = y)$
2. $F_2 = \forall x (f(x) \neq 0)$
3. $F_3 = \forall x (x = 0 \vee \exists y (x = f(y)))$

Give a model with an *uncountable* universe for:

1. $\{F_1, F_2\}$
2. $\{F_1, F_2, F_3\}$

Hint: A set S is uncountable if there is no bijection between S and $\mathbb{N}$.

Solution:

1. $U_{\mathcal{A}} = \mathbb{R}_0^+$, $0^{\mathcal{A}} = 0$, and $f^{\mathcal{A}}(x) = x + 1$
2. We take $U_{\mathcal{A}}$ to be the union of the positive real numbers and the non-positive whole numbers, i.e., $U_{\mathcal{A}} = \mathbb{R}_{>0} \cup \mathbb{Z}_{\leq 0}$. Let $0^{\mathcal{A}} = 0$ and $f^{\mathcal{A}}(x) = 2x$ for $x > 0$ and $f^{\mathcal{A}}(x) = x - 1$ for $x \leq 0$.

Exercise 7.2. [Occurs Check]
What happens if one omits the occurs check in the unification algorithm? Find an example where a unification algorithm without occurs check diverges or returns the wrong result.

Solution:
Consider $x \stackrel{?}{=} f(x)$. Without the occurs check, we first produce $\sigma = \{x \mapsto f(x)\}$. The algorithm keeps going and produces $\sigma' = \{x \mapsto f(f(x))\}$, then $\sigma'' = \{x \mapsto f(f(f(x)))\}$ and so on.

Exercise 7.3. [Unifiable Terms]
Specify the most general unifiers for the following sets of terms, if one exists:

$$L_1 = \{f(x, y), f(h(a), x)\}$$
$$L_2 = \{f(x, y), f(h(x), x)\}$$
$$L_3 = \{f(x, b), f(h(y), z)\}$$
$$L_4 = \{f(x, x), f(h(y), y)\}$$

Solution:

L_1 : $[h(a)/x, h(a)/y]$
L_2 : No unifier, occurs check fails on $x \sim h(x)$
L_3 : $[h(y)/x, b/z]$
L_4 : No unifier, occurs check fails on $h(y) \sim y$

Exercise 7.4. [Formulas without Negation]
Prove that every predicate logic formula that only contains $\wedge, \vee, \forall, \exists, \longrightarrow$ and atomic formulas is satisfiable. Is such a formula also valid?

Solution:
Choose a suitable structure $\mathcal{A}$ that interprets all predicates to be true everywhere. Then, by straightforward induction on the formula, we get that $\mathcal{A}$ is a model.

However, the formula needs not to be valid. Consider, e.g., the formula P for a nullary predicate P. This is clearly not valid, as there are models that interpret P not to hold.

Homework 7.1. [(In)finite Models] (6 points)

1. Show that any model (for a formula of predicate logic) with an universe of size n can be extended to a model of size m for any $m \geq n$. Can it also be extended to an *infinite* model?

2. Now consider the extension of predicate logic with equality. Does above property still hold?

Homework 7.2. [Simultaneous Substitution] (6 points)
Recall that $F[t_1/x_1, \ldots, t_n/x_n]$ is the *simultaneous* substitution of $x_1, \ldots, x_n$ by $t_1, \ldots, t_n$.

1. Can we always express $F[t_1/x_1, \ldots, t_n/x_n]$ as a series of one-variable substitutions?

2. Can we always summarize a series of one-variable substitutions to a single simultaneous substitution?

Homework 7.3. [Most general unifier] (6 points)
Consider the unification problem $x \stackrel{?}{=} y$. Without running the unification algorithm, prove that

1. $\sigma_1 = \{x \mapsto y\}$ is a most general unifier.

2. $\sigma_2 = \{x \mapsto z, y \mapsto z\}$ is unifier, but not a most general unifier.

Hint: Argue using the definition of "most general unifier". Two substitutions σ and σ' can be proven equal by showing that they are equal on all variables, i.e., for all x, $x\sigma = x\sigma'$. Similarly, they can be proven unequal by demonstrating for a particular x that $x\sigma \neq x\sigma'$.

Homework 7.4. [Unification] (2 points)
Use the algorithm presented in the lecture to compute a most general unifier for the following set of formulas: $\{P(x, y), P(f(a), g(x)), P(f(z), g(f(z)))\}$