Logics Exercise

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SS 2017 — Exercise Sheet 10 — 04.07.2017

Submission of homework: Before tutorial on 11.07.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 10.1. [**Sequent Calculus**]
Prove the following formulas in sequent calculus, or give a countermodel that falsifies the formula.

1. $\neg \exists x P(x) \to \forall x \neg P(x)$
2. $(\forall x (P \lor Q(x))) \to (P \lor \forall x Q(x))$
3. $\forall x \exists y P(x,y) \to \exists y \forall x P(x,y)$

Solution:

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(y) \Rightarrow P(y)}}{\Rightarrow P(y), \neg P(y)}\,\neg R}{\Rightarrow \exists x P(x), \neg P(y)}\,\exists R}{\Rightarrow \exists x P(x), \forall x \neg P(x)}\,\forall R}{\neg \exists x P(x) \Rightarrow \forall x \neg P(x)}\,\neg L}{\neg \exists x P(x) \to \forall x \neg P(x)}\,\to R$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\forall x(P \lor Q(x))), P \Rightarrow P, Q(x)}\,Ax \qquad \dfrac{}{(\forall x(P \lor Q(x))), Q(x) \Rightarrow P, Q(x)}\,Ax}{(\forall x(P \lor Q(x))), (P \lor Q(x)) \Rightarrow P, Q(x)}\,\lor L}{\forall x(P \lor Q(x)) \Rightarrow P, Q(x)}\,\forall L}{\forall x(P \lor Q(x)) \Rightarrow P, \forall x Q(x)}\,\forall R}{\forall x(P \lor Q(x)) \Rightarrow P \lor \forall x Q(x)}\,\lor R}{\Rightarrow (\forall x(P \lor Q(x))) \to (P \lor \forall x Q(x))}\,\Longrightarrow R$$

3. $U_{\mathcal{A}} = \{0,1\}, P^{\mathcal{A}} = \{(0,1),(1,0)\}$

Exercise 10.2. [**Counterexamples from Sequent Calculus**]
Consider the following invalid statement: $\exists x P(x) \to \forall x P(x)$. Try to prove this statement in sequent calculus and derive a countermodel from the (incomplete) proof tree.

Exercise 10.3. **[Substitution in Sequent Calculus]**
Prove that $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_G \Gamma[t/x] \Rightarrow \Delta[t/x]$, where, for a set of formulas Γ, we define $\Gamma[t/x]$ to be $\{F[t/x] \mid F \in \Gamma\}$, i.e. free occurrences of x are replaced by t. Give two different proofs:

1. A syntactic proof, transforming the proof tree of $\vdash_G \Gamma \Rightarrow \Delta$.
2. A semantic proof, using correctness and completeness of $\vdash_G$.

Solution:

1. Given a proof of $\Gamma \Rightarrow \Delta$, we can apply the substitution $x \mapsto p$ throughout the proof tree and obtain a valid proof of $\Gamma[x \mapsto p] \Rightarrow \Delta[x \mapsto p]$. Formally, this is an induction on the structure of the proof. Each case in the induction considers one proof rule or axiom, and we have to show that the new proof is still an instance of the axiom. This is trivial in all cases, since the rules are given as schemas that do only concern the structure of the formulas, which is not changed by the substitution.
2. We can also prove the claim by using soundness and completeness. Assume that $\Gamma = \{p_1, \ldots, p_n\}$, $\Delta = \{q_1, \ldots, q_m\}$ and $\Gamma \Rightarrow \Delta$ is derivable. By soundness we know that $p_1 \wedge \cdots \wedge p_n \implies q_1 \vee \cdots \vee q_m$ is a tautology. By the substitution lemma (Corollary 2.4) we know that substituting p for x yields a tautology again, and the completeness theorem asserts that the resulting sequent is provable.

Exercise 10.4. **[Natural Deduction]**
Prove the following formula using natural deduction.

$$\neg(\forall x(\exists y(\neg P(x) \wedge P(y))))$$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(\exists y(\neg P(x) \wedge P(y)))]^1}{\exists y(\neg P(x_1) \wedge P(y))}\,\forall E}{\neg P(x_1) \wedge P(y_1)}\,\exists E}{P(y_1)}\,\wedge E_2 \qquad \dfrac{\dfrac{\dfrac{[\forall x(\exists y(\neg P(x) \wedge P(y)))]^1}{\exists y(\neg P(y_1) \wedge P(y))}\,\forall E}{\neg P(y_1) \wedge P(y_2)}\,\exists E}{\neg P(y_1)}\,\wedge E_1}{\bot}\,\neg E}{\neg(\forall x(\exists y(\neg P(x) \wedge P(y))))}\,\neg I\ (1)$$

Note that a proof step of the following form would *not* be allowed:

$$\dfrac{\forall x(\exists y(\neg P(x) \wedge P(y)))}{\exists y(\neg P(y) \wedge P(y))}\,\forall E$$

Homework 10.1. **[Counterexamples from Sequent Calculus]** (4 points)
Recall Exercise 10.2. We derived a countermodel from an incomplete proof tree. Now consider the statement $\forall x P(x) \to \neg P(x)$.

1. What happens when trying to prove the validity of this formula in sequent calculus?
2. How can we derive a countermodel from the proof tree?
3. Is there a smaller countermodel?

Homework 10.2. **[Proofs]** (16 points)
Prove the following statements using both natural deduction and sequent calculus if they are valid, or give a countermodel otherwise.

1. $\neg\forall x \exists y \forall z(\neg P(x,z) \wedge P(z,y))$
2. $\forall x \forall y \forall z(P(x,x) \wedge (P(x,y) \wedge P(y,z) \to P(x,z)))$
3. $\exists x(P(x) \to \forall x P(x))$

Caution: While you are free to carry out the sequent calculus proofs in Logitext, note that application of $\forall L$ and $\exists R$ delete the principal formula. You have to select “Contract” first before instantiating the principal formula.