Logics Exercise

TU München
Institut für Informatik

Prof. Tobias Nipkow
Lars Hupel

SS 2018 | Exercise Sheet 6 | 15.05.2018

Submission of homework: Before tutorial on 23.05.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 6.1. [Equivalence]
Let F and G be arbitrary formulas. (In particular, they may contain free occurrences of x.) Which of the following equivalences hold? Proof or counterexample!

1. $\forall x(F \wedge G) \equiv \forall x F \wedge \forall x G$
2. $\exists x(F \wedge G) \equiv \exists x F \wedge \exists x G$

Exercise 6.2. [Infinite Models]
Consider predicate logic with equality. We use infix notation for equality, and abbreviate $\neg(s = t)$ by $s \neq t$. Moreover, we call a structure finite iff its universe is finite.

1. Specify a finite model for the formula $\forall x\ (c \neq f(x) \wedge x \neq f(x))$.
2. Specify a model for the formula $\forall x \forall y\ (c \neq f(x) \wedge (f(x) = f(y) \longrightarrow x = y))$.
3. Show that the above formula has no finite model.

Exercise 6.3. [Skolem Form]
Convert the following formula into – in order – a rectified formula, closed and rectified formula, RPF and Skolem form.

$$P(x) \wedge \forall x\ (Q(x) \wedge \forall x \exists y\ P(f(x, y)))$$

Homework 6.1. **[Predicate Logic]** (6 points)

1. Specify a satisfiable formula F such that for all models $\mathcal{A}$ of F, we have $|U_{\mathcal{A}}| \geq 4$. You may or may not use equality.

2. Can you also specify a satisfiable formula F such that for all models $\mathcal{A}$ of F, we have $|U_{\mathcal{A}}| \leq 4$? Consider both predicate logic with and without equality.

Homework 6.2. **[Skolem Form]** (6 points)
Convert the following formulas into – in order – a rectified formula, closed and rectified formula, RPF and Skolem form.

1. $\forall x \exists y \forall z \exists w (\neg Q(f(x), y) \wedge P(a, w))$
2. $\forall z (\exists y (P(x, g(y), z)) \vee \neg \forall x \; Q(x))$

Homework 6.3. **[Orders]** (8 points)
A reflexive and transitive relation is called *preorder*. In predicate logic, preorders can be characterized by the formula

$$F \equiv \forall x \forall y \forall z \; (P(x, x) \wedge (P(x, y) \wedge P(y, z) \rightarrow P(x, z)))$$

1. Which of the following structures are models of F? Give an informal proof in the positive case and a counterexample for the negative case!
   (a) $U^{\mathcal{A}} = \mathbb{N}$ and $P^{\mathcal{A}} = \{(m, n) \mid m > n\}$
   (b) $U^{\mathcal{A}} = \mathbb{Z} \times \mathbb{Z}$ and $P^{\mathcal{A}} = \{((x, y), (a, b)) \mid \; a - x \leq b - y \;\}$
   (c) $U^{\mathcal{A}} = \mathbb{R}$ and $P^{\mathcal{A}} = \{(m, n) \mid m = n\}$
2. Let $Q(x, y)$ be specified as follows: $\forall x \forall y (P(x, y) \leftrightarrow Q(y, x))$. Assuming P is a preorder, is Q also a preorder?
3. Specify the notion of *equivalence relations*, that is, preorders that additionally satisfy symmetry.