Logics Exercise

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Submission of homework: Before tutorial on 10.07.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 13.1. **[Presburger Arithmetic]**
Eliminate the quantifiers from the following formulas, according to Presburger arithmetic:

1. $\forall y(3 < x + 2 \cdot y \vee 2 \cdot x + y < 3)$
2. $\forall x(\exists y(x = 2 \cdot y) \to (2 \cdot x \geq 0 \vee 3 \cdot x < 2))$

Solution:

$$
\begin{aligned}
&\forall y(3 < x + 2 \cdot y \vee 2 \cdot x + y < 3)\\
\leftrightarrow\, &\neg\exists y \neg(3 < x + 2 \cdot y \vee 2 \cdot x + y < 3)\\
\leftrightarrow\, &\neg\exists y(3 \geq x + 2 \cdot y \wedge 2 \cdot x + y \geq 3)\\
\leftrightarrow\, &\neg\exists y(2 \cdot y \leq 3 - x \wedge 3 - 2 \cdot x \leq y)\\
\leftrightarrow\, &\neg\exists y(2 \cdot y \leq 3 - x \wedge 6 - 4 \cdot x \leq 2 \cdot y)\\
\leftrightarrow\, &\neg\exists z(z \leq 3 - x \wedge 6 - 4 \cdot x \leq z \wedge 2 \mid z)\\
\leftrightarrow\, &\neg((6 - 4 \cdot x \leq 3 - x \wedge 2 \mid 6 - 4 \cdot x) \vee (7 - 4 \cdot x \leq 3 - x \wedge 2 \mid 7 - 4 \cdot x))
\end{aligned}
$$

$$
\begin{aligned}
&\forall x(\exists y(x = 2 \cdot y) \to (2 \cdot x \geq 0 \vee 3 \cdot x < 2))\\
\leftrightarrow\, &\forall x(2 \mid x \to (2 \cdot x \geq 0 \vee 3 \cdot x < 2))\\
\leftrightarrow\, &\neg\exists x \neg(2 \mid x \to (2 \cdot x \geq 0 \vee 3 \cdot x < 2))\\
\leftrightarrow\, &\neg\exists x(2 \mid x \wedge 2 \cdot x < 0 \wedge 3 \cdot x \geq 2)\\
\leftrightarrow\, &\neg\exists x(2 \mid x \wedge 2 \cdot x \leq -1 \wedge 2 \leq 3 \cdot x)\\
\leftrightarrow\, &\neg\exists x(2 \mid x \wedge 6 \cdot x \leq -3 \wedge 4 \leq 6 \cdot x)\\
\leftrightarrow\, &\neg\exists z(12 \mid z \wedge z \leq -3 \wedge 4 \leq z) \leftrightarrow \bot
\end{aligned}
$$

Exercise 13.2. **[Quantifier Elimination for $Th(\mathbb{N}, 0, S, =)$]**
Give a quantifier-elimination procedure for $Th(\mathbb{N}, 0, S, =)$ where S is the successor operation on natural numbers, i.e. $S(n) = n + 1$.

Hint: $a = b$ iff $S^k(a) = S^k(b)$ for any $a, b, k \in \mathbb{N}$.

Solution:
We assume $F = \exists x(A_1 \wedge \ldots \wedge A_n)$ where x occurs in all A_i and each A_i is of the form

$$S^k(x) = S^m(t) \text{ or } S^k(x) \neq S^m(t)$$

where t is 0 or a variable (using symmetry of $=$).

If x occurs on both sides of an atom A_i, we can compare the number of successors and replace it with $\bot$ or $\top$. Formally, $Th(\mathbb{N}, 0, S) \models (S^k(x) = S^k(x)) \leftrightarrow \top$ and $Th(\mathbb{N}, 0, S) \models (S^k(x) \neq S^k(x)) \leftrightarrow \bot$, and similarly for different numbers of successors.

Hence, we may further assume that $x \neq t$.

We have to distinguish two cases:

1. All A_i only use $\neq$, but not $=$. Then return $\top$, because

$$Th(\mathbb{N}, 0, S) \models (\exists x(A_1 \wedge \ldots \wedge A_n)) \leftrightarrow \top$$

2. There is at least one A_i of the form $S^m(x) = t$ where $x \neq t$.

   We replace A_i as follows:

   - If $k > 0$, we add the constraints $t \neq 0 \wedge \ldots \wedge t \neq S^{m-1}(0)$ to ensure that the solution for x is non-negative.
   - Otherwise, replace it with $\top$.

   The other A_j $(i \neq j)$ can be replaced as follows: Let A_j be $S^k(x) = u$. Using the hint, first increment both sides by m: $S^{k+m}(x) = S^m(u)$. Then, substitute A_i, resulting in $S^k(t) = S^m(u)$.

   This works similarly for inequality, resulting in $S^k(t) \neq S^m(u)$.

For optimization purposes, we could also assume that either side of the equalities/inequalities contains no successor application. If they do, we can decrement until at least one side is 0 or a variable.

Exercise 13.3. **[Brainstorming]**
Collect about 30 important pieces of terminology that were used in the lecture and that you want to remember for the exam.

Also collect the three results that you find most interesting.

Homework 13.1. **[Quantifier Elimination]** (8 points)
Perform Presburger arithmetic quantifier elimination for each formula:

1. $\forall x \forall y(0 < y \wedge x < y \rightarrow x + 1 < 2 \cdot y)$
2. $\forall x(\exists y(x = 2 \cdot y \wedge 2 \mid y) \rightarrow 4 \mid x)$

Homework 13.2. **[Quantifier Elimination for $Th(\mathbb{Z}, 0, S, P, =, <)$]** (8 points)
Give a quantifier-elimination procedure for $Th(\mathbb{Z}, 0, S, P, =, <)$ where S is the successor and P the predecessor operation on integers, i.e. $S(n) = n + 1$ and $P(n) = n - 1$. Do not use Presburger arithmetic; give a direct algorithm.