

Four Colour Theorem

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- 2 The proof
 - Structure of the proof
 - Simplify the map
 - Consider a minimal counterexample
 - Correct Kempe's mistake
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By Ken Thomson

A student of mine asked me today to give him a reason for a fact which I did not know was a fact - and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any kind of common boundary line are differently coloured - four colours may be wanted, but not more - the following is his case in which four are wanted

A B C D are names of colours



Query cannot a rectangle for 1 from a man be invented so far as I see at this moment, if four compartments have each boundary line in common with one of the others, three of them include the fourth, and prevent any fifth from commencing with it. If this be true, four colours will colour any possible map without any necessity for the colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary A B C two and two - you cannot



make a fourth take boundary from all, except by including one - But it is tricky with acid I am all over of all involutions - What do you say? And how it, if both been added? My pencil says he quipped it in colouring a map of England.



B is included

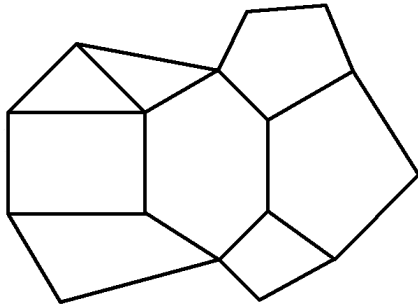
The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Pythons did. If this rule be true the following proposition of logic follows

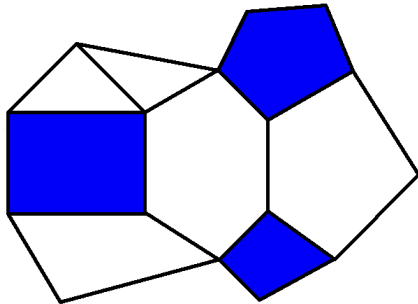
If A B C D be four names of which any two might be conformed by breaking down some well of definition, then some one of the names must be a species of some name which includes within external to the other three

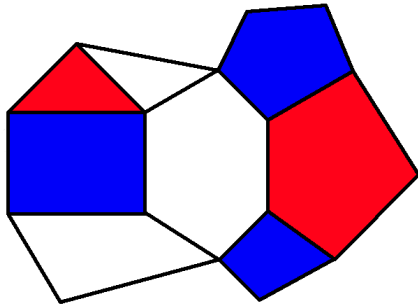
Your truly

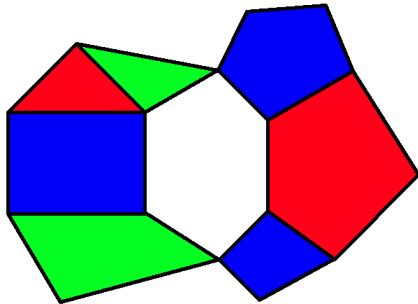
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Theorem (Four Colour Theorem)

The regions of any simple planar map can be coloured with only four colours, in such a way that any two adjacent regions have different colours.

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The regions of any simple planar map can be coloured with only four colours, in such a way that any two adjacent regions have different colours.

Definition (Planar map, regions, simple map)

A *planar map* is a set of pairwise disjoint subsets of the plane, called *regions*. A *simple map* is one whose regions are connected open sets.

Definition (Adjacent regions)

Two regions of a map are *adjacent* if their respective closures have a common point that is not a corner of the map.

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Definition (Corner)

A point is a corner of a map iff it belongs to the closures of at least three regions.

- first proof (attempt) by Kempe in 1879

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- correct proof by Appel and Haken in 1976

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Definition (polygonal outline, polygonal map, face)

A *polygonal outline* is the pairwise disjoint union of a finite number of open line segments, called edges, with a set of nodes that contains the endpoints of the edges. The regions of a *finite polygonal planar map*, called *faces*, are the connected components of the complement of a polygonal outline.

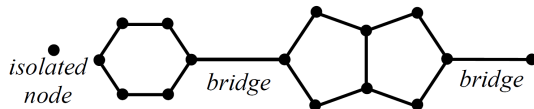


Figure: A general polygonal map

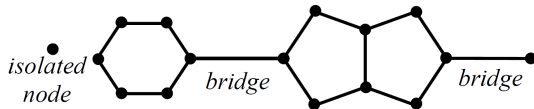


Figure: A general polygonal map

Definition (polyhedral map)

A *polyhedral map* is a finite bridgeless connected polygonal map.

Theorem (Euler formula)

The Euler polyhedron formula (Euler 1752)

$$V - E + F = 2$$

where V, E, F are the number of vertices, edges and faces, is valid for any schema which represents the sphere.

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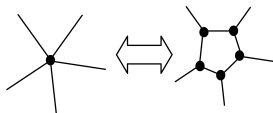


Figure: Reducing to cubic map

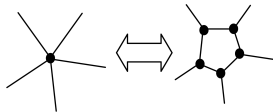


Figure: Reducing to cubic map

$$3V = 2E$$

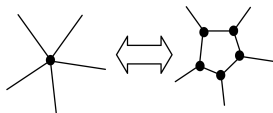


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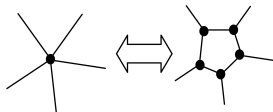


Figure: Reducing to cubic map

$$3V = 2E$$

$$V - E + F = 2$$

$$2E = 6F - 12$$

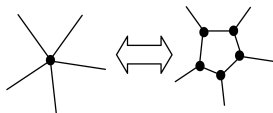


Figure: Reducing to cubic map

$$3V = 2E$$

$$V - E + F = 2$$

$$2E = 6F - 12 \quad \frac{2E}{F} = 6 - \frac{12}{F}$$

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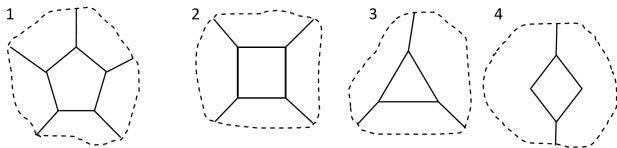
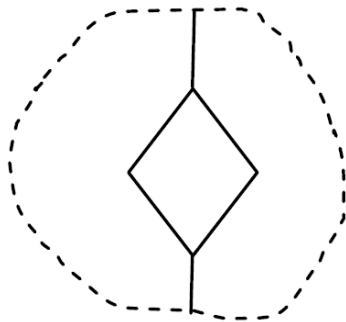
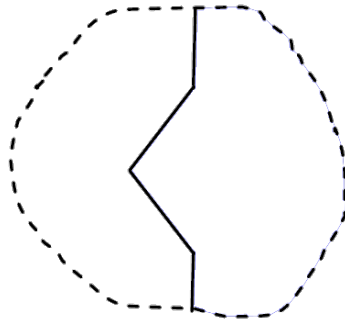
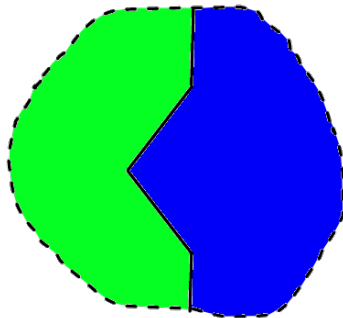
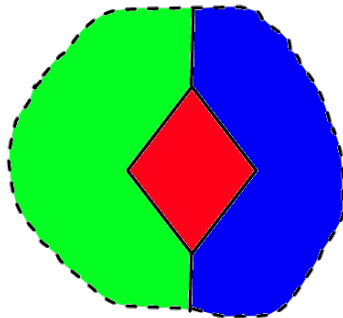


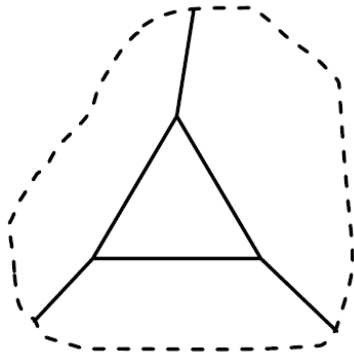
Figure: Possible shape and neighbourhood of the smaller face

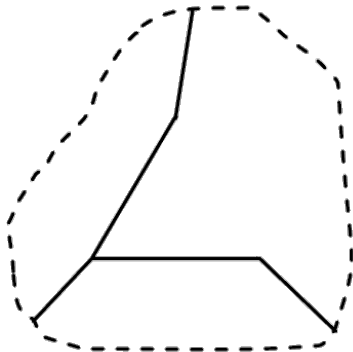


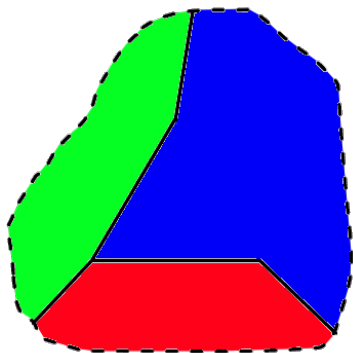


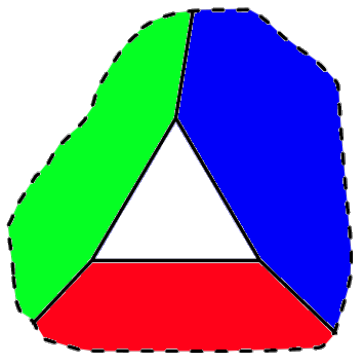


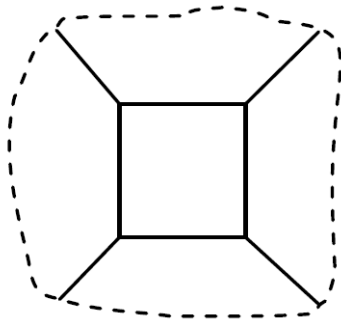


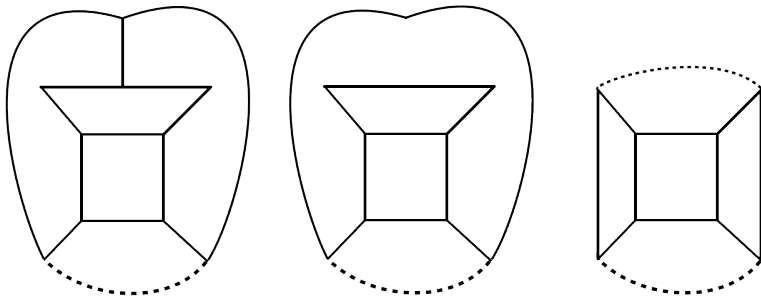


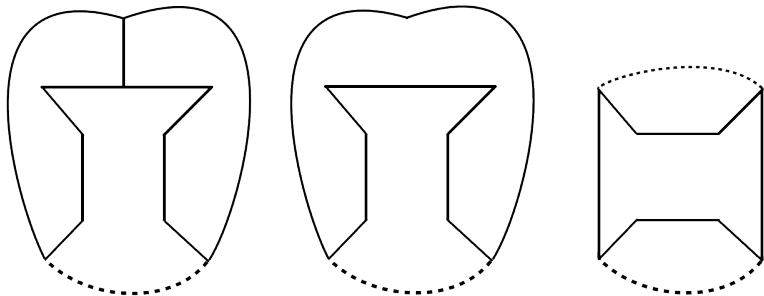


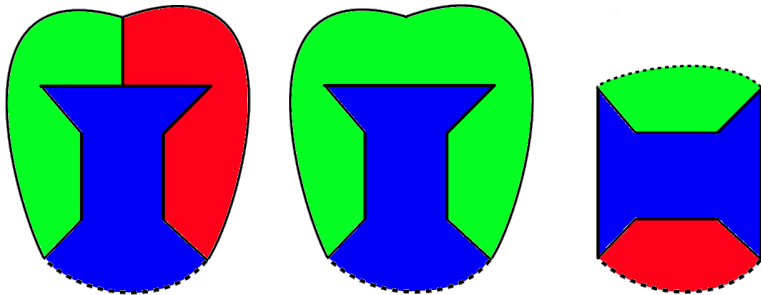


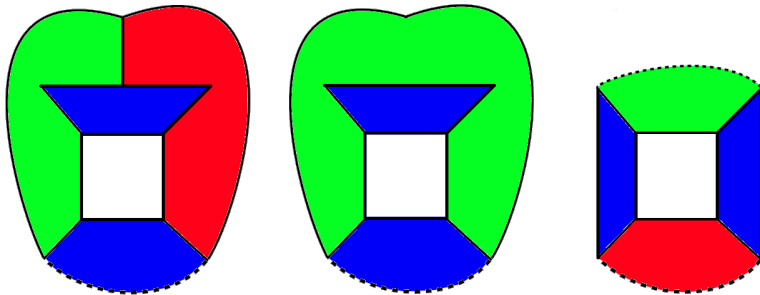


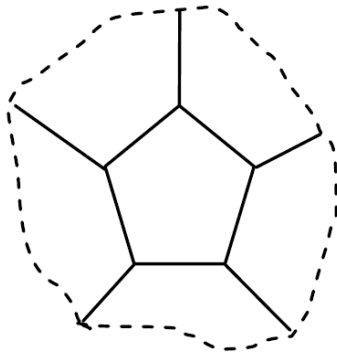












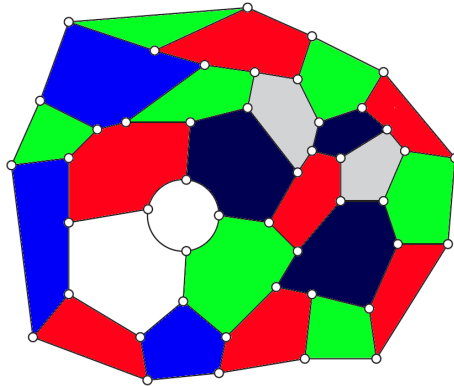


Figure: Kempe chain

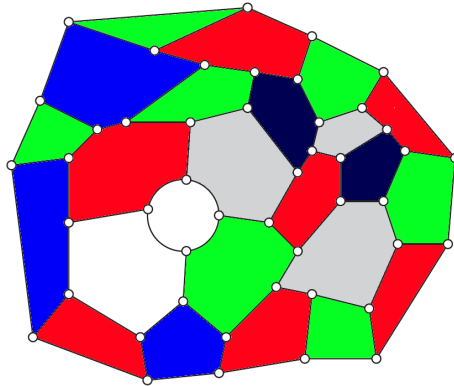


Figure: Kempe chain

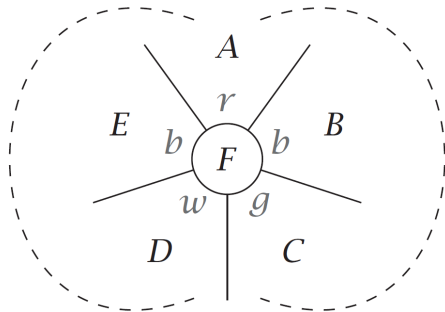


Figure: Kempe chains splitting the colours

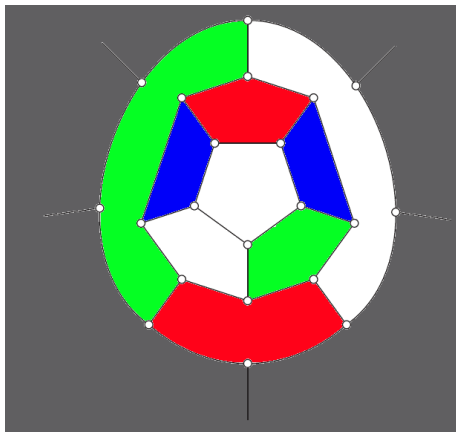


Figure: Counterexample to Kempe's proof

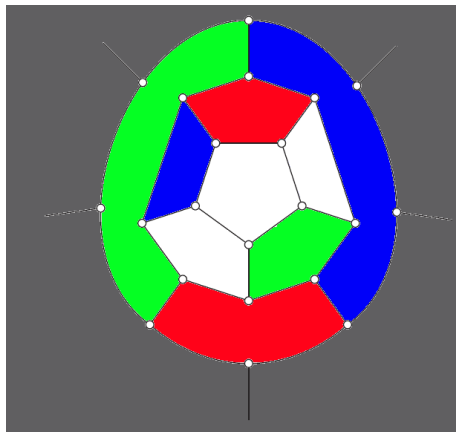


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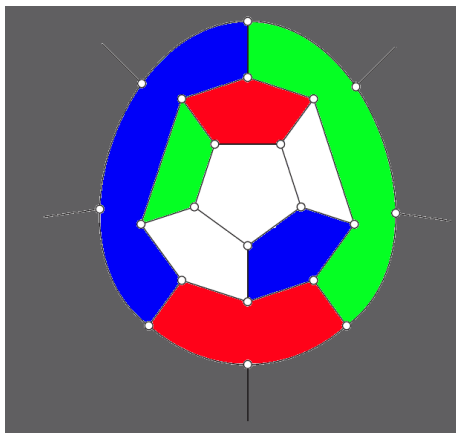


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The task is to find a certain set of configurations that are:

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- *reducible*, meaning that it can be shown that they are 4-colourable and

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- *reducible*, meaning that it can be shown that they are 4-colourable and
- *unavoidable*, meaning that one of these configurations has to appear in every counterexample considered.

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Discharging

$$i(f) = 6 - \#sides(f)$$

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$$\sum_{f \in F} i(f) = 6|F| - 2|E| = 6|F| - 6|F| + 12 = 12$$

Discharge rules

Discharge Rule

Do nothing.

Discharge rules

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Discharge Rule

Transfer $\frac{1}{5}$ from every pentagon to every adjacent face with more than six sides.

Discharge rules

Discharge Rule

Do nothing.

Discharge Rule

Transfer $\frac{1}{5}$ from every pentagon to every adjacent face with more than six sides.

$$r(f_k) = (6 - k) + \frac{m}{5} \leq (6 - k) + \frac{k}{5} = \frac{30 - 5k + k}{5} \leq \frac{-2}{5} < 0$$

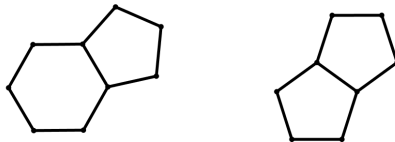


Figure: First unavoidable set

Discharge Rule

Transfer $\frac{1}{4}$ from every pentagon to up to 4 adjacent faces with more than six sides.

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$$r(f_7) = -1 + \frac{m}{4}$$

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Transfer $\frac{1}{4}$ from every pentagon to up to 4 adjacent faces with more than six sides.

$$r(f_7) = -1 + \frac{m}{4}$$

$$r(f_k) = (6 - k) + \frac{m}{4} \leq \frac{24 - 3k}{4} \leq 0$$

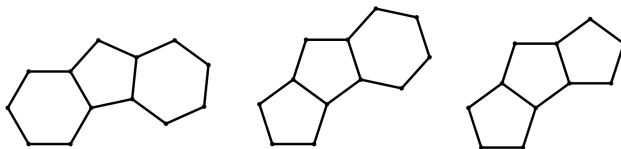


Figure: Second, larger unavoidable set