

The Kepler Conjecture

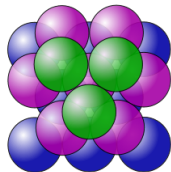
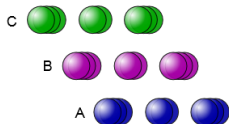
Adrian Rauchhaus

The Theorem

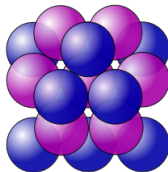
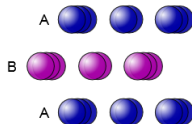
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face-centered cubic (fcc)
(also called cubic close packed (ccp))
three repeating layers ABCABC....



hexagonal close-packed (hcp)
two repeating layers ABABAB...

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- ▶ Fejes Tóth suggests the use of computers for solving ca. 1950
- ▶ Proven by Thomas Hales and Samuel Ferguson in 1998
- ▶ Formalization of the proof in the *FlysPeck* project from 2003 to 2014

The proof assistants

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- ▶ Density is scale invariant \rightarrow Sufficient to consider unit balls
- ▶ Packing can be identified with the centers of the spheres
- ▶ Definition of a packing in HOL Light:

```
|- packing V <=>  
  (!u v. u IN V /\ v IN V /\ dist(u,v) < &2 ==> u = v )
```

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```
|- the_kepler_conjecture <=>
  (!V. packing V
    ==> (?c. !r. &1 <= r
      ==> &(CARD(V INTER ball(vec 0,r))) <=
        pi * r pow 3 / sqrt(&18) + c * r pow 2)
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A large collection of linear programs that are infeasible for the possible counterexamples.

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- ▶ Cells that share critical edges form a *cell cluster*
- ▶ Assign a real number $\Gamma(\epsilon, X)$ to the critical cells, depending on volume, angles between edges and lengths of edges

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$$\forall V \subset A : \sum_{v \in V} f(\|v\|) \leq 12$$

Figure: Constant ball annulus

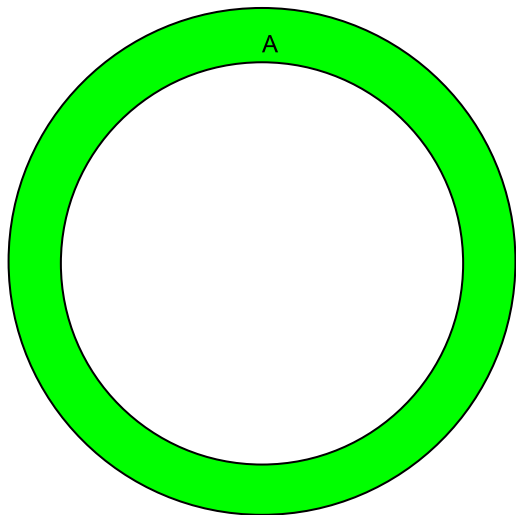
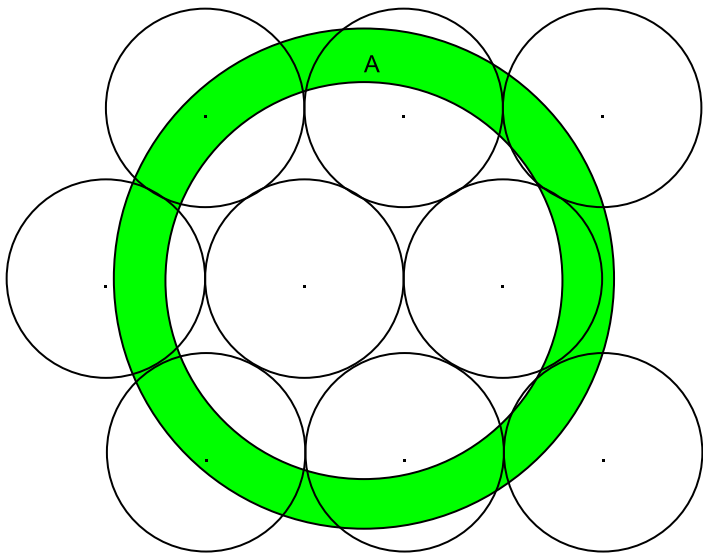


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This leads to the following intermediate result:

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|– the_nonlinear_inequalities /\
  (!V. cell_cluster_inequality V) /\
  (!V. packing V /\ V SUBSET ball_annulus
    ==> local_annulus_inequality V)
  ==> the_kepler_conjecture
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Nonlinear inequalities

- ▶ Most of the inequalities have the form:

$$\forall x, x \in D \Rightarrow f_1(x) < 0 \wedge \dots \wedge f_k(x) < 0$$

with $n \in \mathbb{N}$, $n \leq 6$, $D = [a_1, b_1] \times \dots \times [a_n, b_n]$ and $x = (x_1, \dots, x_n)$

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- ▶ Basic arithmetic operations, square roots, trigonometric functions and the analytic continuation of $\arctan(\sqrt{x})/\sqrt{x}$ to the region $x > -1$

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$$[a_1, b_1] \oplus [a_2, b_2] = [a, b]$$

for some $a \leq a_1 + a_2$ and $b \geq b_1 + b_2$

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- ▶ Other arithmetic operations defined analogously

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Problem:

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These can be verified in about 5000 hours in HOL Light.

Tame classification

Plane graphs:

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- ▶ n-tuples of data including a list of faces
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- ▶ Tame graphs collected in a text file called *archive*
- ▶ Possible counterexamples encoded as tame graphs

Tame classification

Goal:

\vdash " $g \in \text{PlaneGraphs}$ " and "tame g "
implies " $\text{fgraph } g \in_{\simeq} \text{Archive}$ "

fgraph maps graph to the list of faces

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In HOL Light:

```
|- import_tame_classification <==>  
    (!g. g IN PlaneGraphs /\ tame g ==>  
        fgraph g IN_simeq archive)
```

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Isabelle can automatically compute:

$$\vdash \text{fgraph} \text{ ' TameEnum } \subseteq_{\sim} \text{Archive}$$

Tame enumeration

At this point we know:

- ▶ An infinite possible counterexample can be reduced to a finite packing
- ▶ Finite packings can be encoded as plane graphs
- ▶ Only finitely many tame plane graphs exist

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Example: $x \geq \pi \Rightarrow x \geq 3, 14$

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- ▶ This concludes the proof

References

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Pictures:

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