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Semantics of Programming Languages

Exercise Sheet 12

Exercise 12.1 Complete Lattice over Lists

Show that lists of the same length – ordered point-wise – form a partial order if the element type is partially ordered. Partial orders are predefined as the type class *order*.

instantiation *list* :: (*order*) *order*

Define the infimum operation for a set of lists. The first parameter is the length of the result list.

definition *Inf_list* :: *"nat ⇒ ('a::complete_lattice) list set ⇒ 'a list"*

Show that your ordering and the infimum operation indeed form a complete lattice:

interpretation
Complete_Lattice "{xs. length xs = n}" "Inf_list n" **for** *n*

Exercise 12.2 Fixed Point Theory

Let 'a be a complete lattice with ordering $\leq$ and f::$'a \Rightarrow 'a$ be a monotonic function. Moreover, let x_0 be a post-fixpoint of f, i.e., $x_0 \leq f\ x_0$. Prove:

$$\bigsqcup\{f^i(x_0) \mid i \in \mathbb{N}\} \leq \bigsqcup\{f^{i+1}\ (x_0) \mid i \in \mathbb{N}\}$$

Hint: The least upper bound satisfies the following properties

$$x \in A \Longrightarrow x \leq \bigsqcup A \qquad \text{(upper)}$$

$$(\forall\, x \in A.\ x \leq u) \Longrightarrow \bigsqcup A \leq u \qquad \text{(least)}$$

General homework instructions

Both homeworks are pen & paper (or keyboard & text file). You have the choice of uploading a text file or a PDF scan of hand-written notes to the submission system. Physical paper submissions are not accepted.

Homework 12.1 Lattice Theory

Submission until Tuesday, January 23, 2018, 10:00am.

A type $'a$ is a $\sqcup$-semilattice if it is a partial order and there is a supremum operation $\sqcup$ of type $'a \Rightarrow 'a \Rightarrow 'a$ that returns the least upper bound of its arguments:

- Upper bound: $x \leq x \sqcup y$ and $y \leq x \sqcup y$
- Least: $x \leq z \land y \leq z \longrightarrow x \sqcup y \leq z$

Is every finite $\sqcup$-semilattice with a bottom element $\bot$ also a complete lattice? Prove or disprove!

Homework 12.2 Collecting Semantics

Submission until Tuesday, January 23, 2018, 10:00am.

This question concerns the iterative computation of the collecting semantics of the following annotated command where `i` is some positive integer:

```
x := i {A0};
{A1}
WHILE 0 < x
DO {A2} x := x+1 {A3}
{A4}
```

1. Show how the annotations change with each application of the step function. Fill in this table to show the first 7 steps of the process:

	0	1	2	3	4	5	6	7
$A0$	$\{\}$	$\{i\}$						
$A1$	$\{\}$							
$A2$	$\{\}$							
$A3$	$\{\}$							
$A4$	$\{\}$							

   For brevity, abbreviate a state $<x := k>$ by the value k of x when you fill in the table. Entries that do not change can be left blank.

2. What are the annotations $A0, \ldots, A4$ of the collecting semantics of the above annotated command, i.e. of the least fixpoint of function *step*?